Appendix of "Efficient Non-domination Level Update Method for Steady-State Evolutionary Multi-objective Optimization"

Appendix A
Proof of Lemma 1

Proof: First of all, let us give some terminologies used in this proof.

- d_i: The number of solutions dominated by the newly added solutions in F_i, where $i \in \{1, \cdots, l\}$ and $0 \leq d_i \leq \varphi_i$.
- indicator function $I_A(z) = \begin{cases} 1 & \text{if } z \geq 0 \\ 0 & \text{otherwise} \end{cases}$.
- indicator function $I_B(F_i)$[1].

Let $\tau_i = \varphi_i - d_i - 1$ and $\omega_i = [1 - I_B(F_i)]d_i + I_B(F_i)$, the mathematical model of NoC is formulated as follows:

$$NoC = \varphi_1 + \sum_{i=1}^{l-1}[\prod_{j=1}^{i} I_A(\tau_j)I_A(\omega_j)]\omega_i\varphi_{i+1} \tag{1}$$

According to this model, we can easily find that NoC is constantly equals φ_1, i.e., N, when $l = 1$. When $l \geq 2$, we use induction to finalize our proof.

- When $l = 2$, according to equation (1), we have

$$NoC = \varphi_1 + I_A(\tau_1)I_A(\omega_1)\omega_1\varphi_2 \tag{2}$$

  where $\tau_1 = \varphi_1 - d_1 - 1$ and $\omega_1 = [1 - I_B(F_1)]d_1 + I_B(F_1)$. Obviously, $NoC = \varphi_1$ when the newly added offspring either dominates or is non-dominated with all solutions in F_1. Otherwise, in order to maximize NoC, we should satisfy the following three conditions:

  1) $I_A(\tau_1) = 1$: it means that $\tau_1 > 0$.
  2) $d_1 > 0$: it means that the newly added offspring dominates some solutions in F_1.
  3) $I_B(F_1) = 0$: it means that the newly added offspring are non-dominated with some solutions in F_1.

  In this case, NoC is maximized when $d_1 = \varphi_1 - 1$, which means that equation (2) of our paper is true for $l = 2$.
- Suppose equation (2) of our paper is true for $l = n$, then when $l = n + 1$, we have:

$$NoC = NoC_n + \prod_{j=1}^{n} I_A(\tau_j)I_A(\omega_j)\omega_n\varphi_{n+1} \tag{3}$$

  where $NoC_n = \varphi_1 + \sum_{i=1}^{k}(\varphi_{i-1} - 1)\varphi_i$. $k = n$ in case there does not exist any NDL, before F_n, in which the newly added solutions from the previous NDL dominates or are non-dominated with all solutions; otherwise k is the index of the first such NDL. Accordingly, we will have $NoC = NoC_n$ in case there exists a NDL, before F_l, in which the newly added solutions from the previous NDL dominates or are non-dominated with all solutions. This is because the production term in equation (3) will be zero in that case. On the flip side, similar to the proof when $l = 2$, we also need to satisfy the following three conditions:

  1) $I_A(\tau_n) = 1$: it means that $\tau_n > 0$.
  2) $d_n > 0$: it means that the newly added solutions, moved from F_{n-1} to F_n, dominates some solutions in F_n.
  3) $I_B(F_n) = 0$: it means that the newly added solutions, moved from F_{n-1} to F_n, are non-dominated with some solutions in F_n.

In this case, we can maximize NoC when $d_n = \varphi_n - 1$. All in all, equation (2) of our paper holds for $l = n + 1$, and the proof of the induction step is completed. ∎

[1] $I_B(F_i) = \begin{cases} 0 & \text{if added solutions are non-dominated with those in } F_i. \\ 1 & \text{otherwise} \end{cases}$

Appendix B
Proof of Lemma 2

Proof: According to Lemma 1, when $l = 2$ the largest NoC is calculated as:

$$NoC = \varphi_1 + (\varphi_1 - 1)\varphi_2 \tag{4}$$

Since $\varphi_1 + \varphi_2 = N$, we use $\varphi_2 = N - \varphi_1$ to do substitution in equation (4):

$$NoC = -\varphi_1^2 + (N + 2)\varphi_1 - N \tag{5}$$

To find the value of φ_1 that maximizes NoC, we take the derivative of φ_1 on both sides of equation (5). This process gives us $\varphi_1 = [\frac{N}{2}] + 1$ and $\varphi_2 = N - [\frac{N}{2}] - 1$. Note that $[*]$ can either be a rounded up or rounded down operation in case $\frac{N}{2}$ is not an integer. ■

Appendix C
Proof of Lemma 3

Proof: According to Lemma 1, we can unfold the equation (2) of our paper and rewrite it as follows:

$$\begin{aligned} NoC &= \varphi_1 + \sum_{i=2}^{k}(\varphi_{i-1}\varphi_i - \varphi_i) \\ &= \varphi_1(\varphi_2 + 1) + \sum_{i=2}^{k-1}\varphi_i(\varphi_{i+1} - 1) - \varphi_k \end{aligned} \tag{6}$$

where $\varphi_i \geq 1$, $i \in \{1, \cdots, k\}$, $\sum_{i=1}^{l}\varphi_i = N$ and $l \geq k$. Note that there is only one subtraction term, i.e., $-\varphi_k$, in equation (6). Assume that φ_i, $i \in \{1, \cdots, k\}$, are unknown to us. If we decrease φ_k by 1 and increase φ_{k-2} by 1 accordingly, which is used to meet the constraint $\sum_{i=1}^{l}\varphi_i = N$, the resulting NoC' is calculated as:

$$\begin{aligned} NoC' = \varphi_1(\varphi_2 + 1) + \sum_{i=2}^{k-4}\varphi_i(\varphi_{i+1} - 1) + \varphi_{k-3}(\varphi'_{k-2} - 1) \\ + \varphi'_{k-2}(\varphi_{k-1} - 1) + \varphi_{k-1}(\varphi'_k - 1) - \varphi'_k \end{aligned} \tag{7}$$

where $\varphi'_{k-2} = \varphi_{k-2} + 1$ and $\varphi'_k = \varphi_k - 1$. Then, equation (7) can be further written as:

$$\begin{aligned} NoC' = \varphi_1(\varphi_2 + 1) + \sum_{i=2}^{k-4}\varphi_i(\varphi_{i+1} - 1) + \varphi_{k-3}(\varphi_{k-2} - 1) \\ + \varphi_{k-3} + \varphi_{k-2}(\varphi_{k-1} - 1) + \varphi_{k-1}(\varphi_k - 1) - \varphi_k \end{aligned} \tag{8}$$

Obvioulsy, we have $NoC' - NoC = \varphi_{k-3} \geq 1$. If this above procedure iterates until φ_k is decreased to 1, the resulting NoC will be increased by $(\varphi_k - 1)\varphi_{k-3}$ at the end. Let us rewrite equation (6) as:

$$NoC = \varphi_1(\varphi_2 + 1) + \sum_{i=2}^{k-2}\varphi_i(\varphi_{i+1} - 1) - 1 \tag{9}$$

where $\varphi_i \geq 1$, $i \in \{1, \cdots, k\}$, $\varphi_k = 1$, $\sum_{i=1}^{l}\varphi_i = N$ and $l \geq k$. If we decrease φ_{k-1} by 1 and increase φ_{k-3} by 1 accordingly, which is used to meet the constraint $\sum_{i=1}^{l}\varphi_i = N$, the resulting NoC' is calculated as:

$$\begin{aligned} NoC' = \varphi_1(\varphi_2 + 1) + \sum_{i=2}^{k-5}\varphi_i(\varphi_{i+1} - 1) + \varphi_{k-4}(\varphi'_{k-3} - 1) \\ + \varphi'_{k-3}(\varphi_{k-2} - 1) + \varphi_{k-2}(\varphi'_{k-1} - 1) - 1 \end{aligned} \tag{10}$$

where $\varphi'_{k-3} = \varphi_{k-3} + 1$ and $\varphi'_{k-1} = \varphi_{k-1} - 1$. Then, equation (10) can be further written as:

$$\begin{aligned} NoC' = \varphi_1(\varphi_2 + 1) + \sum_{i=2}^{k-5}\varphi_i(\varphi_{i+1} - 1) + \varphi_{k-4}(\varphi_{k-3} - 1) \\ + \varphi_{k-3}(\varphi_{k-2} - 1) + \varphi_{k-2}(\varphi_{k-1} - 1) - 1 + (\varphi_{k-4} - 1) \end{aligned} \tag{11}$$

Note that $\varphi_{k-4} \neq 1$, otherwise there is no further comparison after F_{k-4} according to Lemma 1. Therefore, according to equation (10) and equation (11), we have $NoC' - NoC = \varphi_{k-4} - 1 > 0$. Similar to the previous derivation, if this above procedure iterates until φ_{k-1} is decreased to 1, the resulting NoC will be increased by $(\varphi_{k-1} - 1)(\varphi_{k-4} - 1)$ at the end.

In fact, if we repeat the above derivation, we can have:

$$NoC = \varphi_1(\varphi_2 + 1) - 1 \tag{12}$$

where $\varphi_1 > 1$, $\varphi_2 > 1$, $\varphi_i = 1$, $i \in \{3, \cdots, k\}$, $\sum_{i=1}^{l} \varphi_i = N$ and $l \geq k$. Using $\varphi_2 = N - l + 2 - \varphi_1$ to do substitution in equation (12):

$$NoC = -\varphi_1^2 + (N - l + 3)\varphi_1 - 1 \tag{13}$$

Similar to the proof in Appendix B, we take the derivative of φ_1 on both side of equation (13) to find the value of φ_1 that maximizes NoC. This process gives us $\varphi_1 = [\frac{N-l+3}{2}]$, $\varphi_2 = N - [\frac{N-l+3}{2}] - 1$, where $[*]$ can either be a rounded up or rounded down operation in case $\frac{N-l+3}{2}$ is not an integer. ■

Appendix D
Proof of Theorem 1

Proof: Lemma 1 and Lemma 3, for a given N and some $l \geq 3$, we have:

$$NoC = \varphi_1 + \varphi_1\varphi_2 - 1 \tag{14}$$

where $\varphi_1 = [\frac{N-l+3}{2}]$, $\varphi_2 = N - [\frac{N-l+3}{2}] - 1$ and $\varphi_i = 1$, $i \in \{3, \cdots, l\}$. Obviously, in order to maximize NoC in equation (14), we need to make $l = 3$ so that φ_1 and φ_2 do not need to sacrifice points to the other residual φ_is. In other words, for a given N, $l = 3$ (when $l \geq 3$) maximizes NoC and the corresponding NDL structure is $\varphi_1 = [\frac{N}{2}]$, $\varphi_2 = N - [\frac{N}{2}] - 1$ and $\varphi_3 = 1$, where $[*]$ can either be a rounded up or rounded down operation in case $\frac{N}{2}$ is not an integer. Using these φ_is, $i \in \{1, 2, 3\}$, in equation (14), we have the maximum NoC, denoted as NoC_{max}, is calculated as:

$$NoC_{max} = -[\frac{N}{2}]^2 + [\frac{N}{2}]N - 1 \tag{15}$$

On the other hand, according to Lemma 1 and Lemma 2, we have the maximum NoC when $l = 2$, denoted as NoC'_{max}, is calculated as:

$$NoC'_{max} = -[\frac{N}{2}]^2 + [\frac{N}{2}]N + 1 \tag{16}$$

Obviously, $NoC'_{max} > NoC_{max}$. ■

Appendix E
Further Investigations of Non-domination Level (NDL) Structure

In this section, we investigate some interesting properties of the NDL structure. As discussed in [1], for a randomly generated population, the number of NDLs diminishes with the increase of the number of objectives. To validate this assertion, we conduct an experiment on several randomly generated populations. The population size is set as $N = 200$, $N = 800$ and $N = 2,000$, respectively. Each point in a population is randomly sampled from a uniform distribution within the range $\prod_{i=1}^{m}[0, 1]$. The number of objectives varies from 2 to 15 with an increment of one. As shown in Fig. 1, all three trajectories drop rapidly with the growth of dimensionality. In addition, we also notice that the deterioration of the number of NDLs is more significant in case the population size is small.

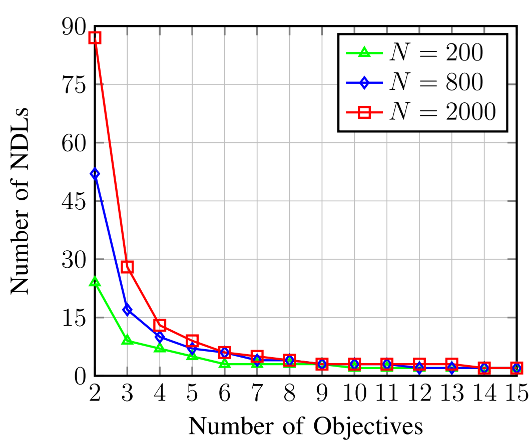

Fig. 1: Number of NDLs formed in a random population, with population size 200, 800 and 2,000, respectively, for different number of objectives.

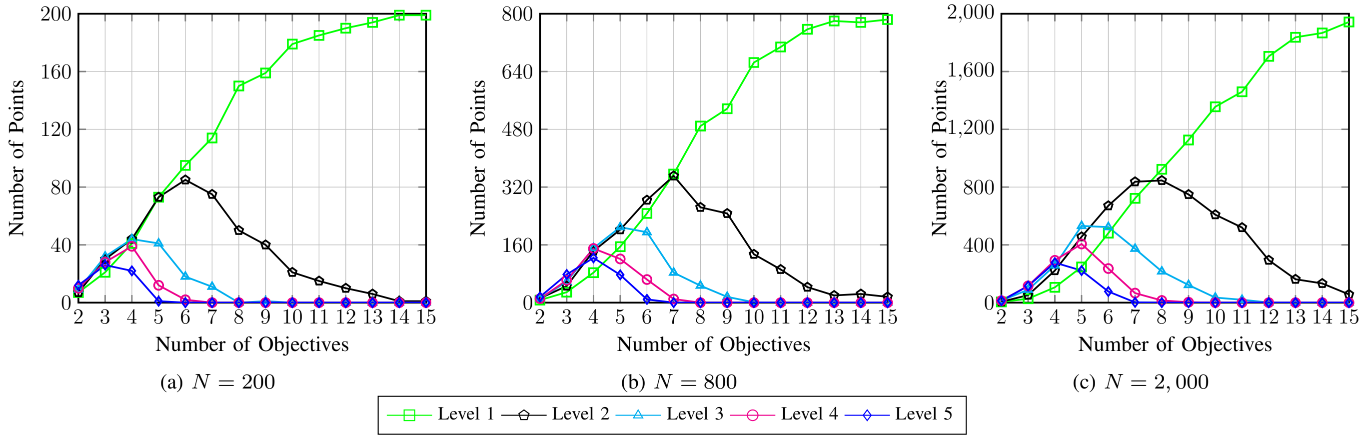

Fig. 2: Number of points in the first five NDLs for different number of objectives.

In real optimization scenarios, decision maker might be more interested in the amount of solutions in the first few NDLs. Here, we conduct another experiment to investigate the variation of the amount of solutions in the first five NDLs for different number of objectives. From the results shown in Fig. 2, we find that the trajectories for different population sizes share a similar trend. Specifically, the amount of solutions in the first NDL steadily increases with the growth of dimensionality, while for the other four NDLs, the amount of solutions slightly increases at first but rapidly decreases later on. Note that this observation conforms to the previous experiment where the number of NDLs decreases with the growth of dimensionality.

The experiments on a randomly generated population demonstrate some interesting properties of the NDL structure in the early stage of evolution. It is also interesting to investigate the scenarios in the late stage of evolution, where the population almost approaches the Pareto-optimal front. To this end, we design another synthetic data set where each sample point is within a band region right upon the Pareto-optimal front of DTLZ2 [2]. Fig. 3 presents a simple example of 800 such randomly sampled points in a three-dimensional space. More specifically, the mathematical formulation of DTLZ2 is as follows:

$$\begin{aligned}
f_1(\mathbf{x}) &= (1+g(\mathbf{x}_m))\cos(\frac{x_1\pi}{2})\cdots\cos(\frac{x_{m-2}\pi}{2})\cos(\frac{x_{m-1}\pi}{2})\\
f_2(\mathbf{x}) &= (1+g(\mathbf{x}_m))\cos(\frac{x_1\pi}{2})\cdots\cos(\frac{x_{m-2}\pi}{2})\sin(\frac{x_{m-1}\pi}{2})\\
f_3(\mathbf{x}) &= (1+g(\mathbf{x}_m))\cos(\frac{x_1\pi}{2})\cdots\sin(\frac{x_{m-2}\pi}{2})\\
&\cdots\\
f_m(\mathbf{x}) &= (1+g(\mathbf{x}_m))\sin(\frac{x_1\pi}{2})
\end{aligned}$$

where $\mathbf{x}\in\Omega=\prod_{i=1}^{n}[0,1]$ and $g(\mathbf{x}_m)=\sum_{x_i\in\mathbf{x}_m}(x_i-0.5)^2$. To obtain a sample point as shown in Fig. 3, we set x_i, where $i\in\{1,2,\cdots,m-1\}$, be sampled from the range $[0,1]$ and x_j, where $j\in\{m,m+1,\cdots,n\}$, be sampled from the range $[0.5,0.9]$. Similar to the previous experiments on the randomly generated population, we also investigate two aspects, i.e., the variation of the number of NDLs in different dimensionality and the variation of the amount of solutions in the first five NDLs.

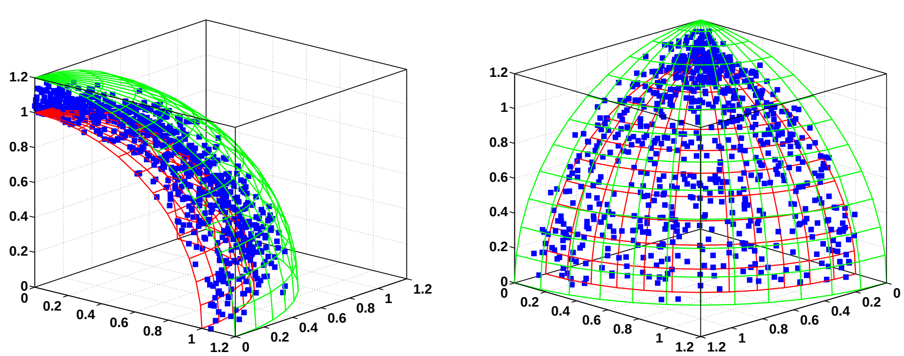

Fig. 3: 800 randomly sampled points within a band region for a three-dimensional case.

From the experimental results shown in Fig. 4 and Fig. 5, we find the trajectories share a similar trend as those in Fig. 1 and Fig. 2. However, it is also worth noting that the number of NDLs formed in this synthetic data set is much fewer than that in the randomly generated population. This observation implies that the number of NDL structure becomes relatively simpler when the population almost converges to the Pareto-optimal front.

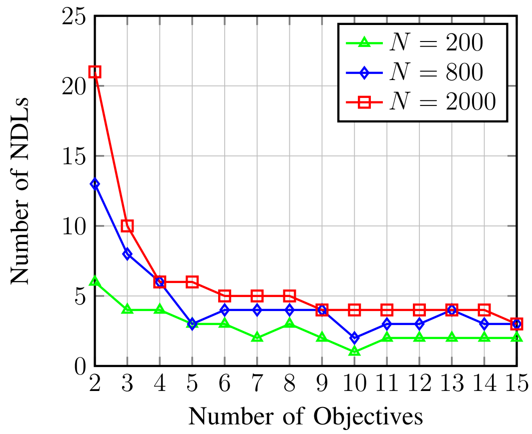

Fig. 4: Number of NDLs formed in the synthetic data sets with population size 200, 800 and 2,000, respectively, for different number of objectives.

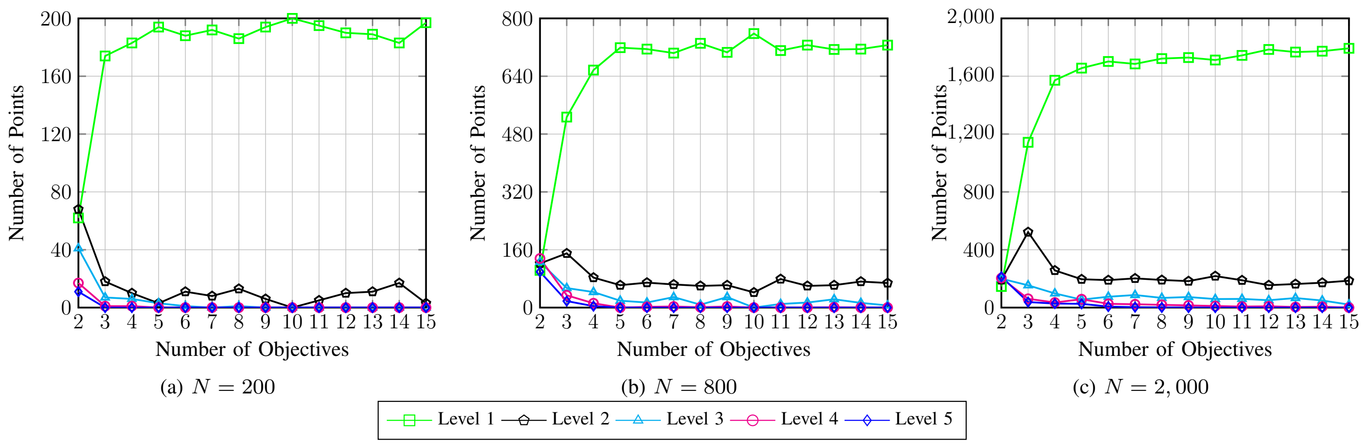

Fig. 5: Number of points in the first five NDLs for different number of objectives.

References

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[2] K. Deb, L. Thiele, M. Laumanns, and E. Zitzler, "Scalable test problems for evolutionary multiobjective optimization," in *Evolutionary Multiobjective Optimization*, ser. Advanced Information and Knowledge Processing, A. Abraham, L. Jain, and R. Goldberg, Eds. Springer London, 2005, pp. 105–145.